JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 17, No. 2, March–April 1994

New Output Feedback Design in Variable Structure Systems

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This paper proposes an output feedback variable structure control to stabilize a class of uncertain systems in which the state is unavailable and no estimated state is required. The special sliding hyperplane is introduced so that the output of the system is initially on the hyperplane no matter where the initial output is, and the stability of the equivalent reduced-order system in the sliding mode is assured under a certain condition. Further, based on the concept of the equivalent motion and the known bound of the initial state, the constant control gain is derived to guarantee the existence of the sliding mode. Finally, an aircraft model is given to illustrate this design approach.

I. Introduction

VARIABLE structure systems (VSS) are a special class of nonlinear systems characterized by a discontinuous control action which changes the system structure on the state reaching the sliding (switching) hyperplane. The major merit of the VSS is their insensitivity to parameter variations and external disturbances.[1,2] Hence, over the last few years, the VSS approach has been widely applied to the design of practical control systems, such as servo systems,[3–5] power systems,[6,7] flight control systems,[8] etc. However, the conventional VSS are always limited to the systems with full-state feedback. In fact, in practical application, full measurement of state might be neither possible nor feasible. Recently, some asymptotic observers and dynamic compensators have been used in VSS to deal with the unavailability of state,[9–11] even though they possibly increase the complexity of the system. Therefore, the direct output feedback design in VSS is worth investigating. Up to now, there has been very little relevant literature addressing this subject. Heck and Ferri[12] proposed a direct output feedback in VSS by choosing a matrix N such that the nominal system satisfies the reaching condition; nevertheless, the existence of the matrix N is not guaranteed.

In this paper, we proposed a direct output feedback variable structure control to stabilize the uncertain system robustly. Here the state is unavailable and no estimated state is required. The main idea is to choose a special set of switching functions and place the system on the sliding hyperplane at the initial instant; thereafter, the existence condition of the sliding mode must be guaranteed. With the aid of the known bound of the initial state and the concept of the equivalent motion, the constant control gain is derived to ensure the existence of the sliding mode.

The organization of this paper is as follows: In Sec. II we formulate the system and state the problem. The main results are derived in Sec. III. In Sec. IV an illustrated example is given. A conclusion is given in the last section.

II. System Formulation and Problem Statement

Consider a class of uncertain systems described as

$$\dot{\bar{x}} = (\bar{A} + \Delta\bar{A})\bar{x} + \bar{B}u \tag{1a}$$

$$y = \bar{C}\bar{x} \tag{1b}$$

$$\Delta\bar{A} = \bar{B}\bar{D} \tag{1c}$$

where the state vector $\bar{x} \in R^n$, and the control input $u \in R^m$, and the output vector $y \in R^p$, for $m \le p \le n$. $(\bar{A}, \bar{B})$ is a completely controllable pair with appropriate dimensions. Moreover, the matrix $\bar{A}$ represents the nominal linear part of the system, $\bar{B}$ is the input matrix of full rank and $\Delta\bar{A}$ is a matrix involving all possible system parameter variations satisfying the matching condition Eq. (1c). Let Eq. (1) be transformed as follows[13,14]:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} M\bar{x}, \qquad x_1 \in R^{n-m}, \quad \text{and} \quad x_2 \in R^m \tag{2}$$

where M is an $n \times n$ orthogonal transformation matrix such that

$$M\bar{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \equiv B \tag{3}$$

where $B_2 \in R^{m\times m}$ is a nonsingular matrix. Then Eq. (1) can be written as

$$\dot{x} = (A + \Delta A)x + Bu \tag{4a}$$

$$y = Cx \tag{4b}$$

where

$$A = M\bar{A}M^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$\Delta A = M\Delta\bar{A}M^{-1} = BD, D = \bar{D}M^{-1} \tag{4c}$$

$$\bar{C} = \bar{C}M^{-1} = [C_1 \quad C_2], \quad C_1 \in R^{p\times(n-m)} \quad \text{and} \quad C_2 \in R^{p\times m}$$

in which we assume that the uncertainty matrix D has the following bound

$$\|D\|_2 \le d \tag{4d}$$

where $\|D\|_2$ denotes the spectral norm of matrix D and d is a positive scalar. In the following, without loss of generality, our analysis and design are on the basis of the transformed equivalent system Eq. (4).

Our main objective is to design an output feedback control such that the system in Eq. (4) is stabilized robustly. The components of

Received Sept. 1, 1992; revision received May 15, 1993; accepted for publication June 16, 1993.

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the control vector $u \in R^m$ are of variable structure form, i.e.,

$$u_i = \begin{cases} u_i^+(y) & \text{if} \quad s_i(y) > 0 \\ u_i^-(y) & \text{if} \quad s_i(y) < 0 \end{cases}, \quad i = 1, 2, \dots, m \tag{5a}$$

where

$$s_i(y) \equiv g_i[y(t) - \exp(-\beta_i t)\, y(0)], \qquad g_i \in R^{1\times p} \tag{5b}$$

is the ith switching function, each β_i is a positive scalar, and $y(0)$ is the initial output. For simplicity, choosing $\beta_1 = \dots = \beta_m = \beta > 0$. Let us define a matrix $G \in R^{m\times p}$; each row of matrix G is g_i, $i = 1, 2, \dots, m$, which will be determined later. The sliding mode occurs when the output reaches and remains on the intersection of the m switching functions $s_i = 0$, $i = 1, 2, \dots, m$, i.e.,[15]

$$S(y) = [s_1, \dots, s_m]^T = G[y - \exp(-\beta t)y(0)] = 0 \tag{6}$$

$S(y) = 0$ is so called the "sliding hyperplane". $S[y(0)] = 0$ means that the system is placed on the sliding hyperplane initially no matter where $y(0)$ is. Besides, the exponential term $\exp(-\beta t)y(0)$ will decay to zero as $t \to \infty$, and its effect will be discussed later. Here we assume GC_2, and GCB are both nonsingular matrices.

III. Main Result

The design procedure can be divided into two steps as follows.

A. Sliding Hyperplane Determination

While in the sliding mode, i.e., $S = 0$, a certain linear dependence among the state is as below

$$x_2 = -(GC_2)^{-1}GC_1x_1 + (GC_2)^{-1}Gy(0)\exp(-\beta t) \tag{7a}$$

thus the system in Eq.(4) can be reduced to the following n-m dimensional form

$$\dot{x}_1 = [A_{11} - A_{12}(GC_2)^{-1}GC_1]x_1 + A_{12}(GC_2)^{-1}Gy(0)\exp(-\beta t) \tag{7b}$$

and

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} I \\ -(GC_2)^{-1}GC_1 \end{bmatrix} x_1(t) + \begin{bmatrix} 0 \\ -(GC_2)^{-1}Gy(0) \end{bmatrix} \exp(-\beta t) \tag{7c}$$

where $I \in R^{(n-m)\times(n-m)}$ is an identity matrix. For convenience, the following notations are used in the sequel of this paper:

$$\hat{A}_1 \equiv A_{11} - A_{12}(GC_2)^{-1}GC_1 \tag{8a}$$

$$\phi(t) \equiv A_{12}(GC_2)^{-1}Gy(0)\exp(-\beta t) \tag{8b}$$

and

$$H \equiv \begin{bmatrix} I \\ -(GC_2)^{-1}GC_1 \end{bmatrix} \in R^{n\times(n-m)} \tag{8c}$$

The sliding hyperplane $S(y) = 0$ is determined by choosing the matrix G to stabilize the reduced-order system in the sliding mode. Neglecting the exponential term ϕ first, Eq. (7b) can be seen as a usual linear output feedback problem

$$\dot{x}_1 = [A_{11} - A_{12}KC_1]x_1 \tag{9}$$

where $K = (GC_2)^{-1}G$. The following lemma is needed.

Lemma 1[12,16]: For a matrix $C_2 \in R^{n\times m}$, there exists a matrix $G \in R^{m\times p}$ such that $K = (GC_2)^{-1}G$ in Eq. (9) can achieve some poles placement if and only if

$$\operatorname{rank}[C_2K - I] \le p - m \tag{10}$$

That is, if Eq. (10) holds, we can select K to stabilize the system in Eq. (9) first, and there exists a suitable G to satisfy $G(C_2K - I) = 0$. Here the solution G may not be unique.

Remark 1: It is noted that the exponential term ϕ will decay to zero as $t \to \infty$, hence it will not affect the equilibrium point of the system Eq. (7b). That is, the sliding hyperplane in Eq. (6) will converge to the following form:

$$\lim_{t\to\infty} S(t) = \lim_{\phi\to 0} S(t) = Gy(t)$$

so the preceding determination of the matrix G by neglecting ϕ is reasonable. Besides, the response prior to this convergence might be not very good. The decay rate of ϕ can be speeded up by increasing the value of β, however, the control gain will increase too.

B. Existence Condition Satisfaction

It is well-known that the output can globally reach the sliding hyperplane if[1,2]

$$S^T(y)\dot{S}(y) < 0 \tag{11a}$$

Because the system is on the sliding hyperplane initially, we only need to consider the existence condition of the sliding hyperplane, i.e.,

$$\lim_{s\to 0} S^T(y)\dot{S}(y) < 0 \tag{11b}$$

To achieve the existence condition in Eq. (11b), the following approach is helpful.

Control Selection: For the given system in Eq. (4) with known bound of initial state $x(0)$, let the control $u(y)$ be selected as

$$u(y) = -k(GCB)^{-1}\operatorname{sgn}(S) \tag{12}$$

where $\operatorname{sgn}(S) = [\operatorname{sgn}(s_1), \dots, \operatorname{sgn}(s_m)]^T$, each $\operatorname{sgn}(s_i)$ is a signum function of s_i, $i = 1, 2, \dots, m$, defined as $\operatorname{sgn}(s_i) = 1$ for $s_i > 0$, 0 for $s_i = 0$, -1 for $s_i < 0$. Let the constant control gain $k \in R$ satisfy the following inequality

$$k \ge \max[(\|GCA\|_2 + \|GC\|_2\|\Delta A\|_2)L + \beta\|Gy(0)\|_2] \tag{13}$$

where $\max(\cdot)$ represents the maximum value of $(\cdot)$ and L is defined as

$$L = \|H\|_2\left\{\|\exp[\hat{A}_1(t)]\|_2\|x(0)\|_2 + \left\|\int_0^t \exp[\hat{A}_1(t-\tau)]\phi(\tau)\,d\tau\right\|_2\right\} + \left\|(GC_2)^{-1}Gy(0)\right\|_2 \tag{14}$$

for $t \ge 0$, then Eq. (11b) holds, i.e., the existence condition is guaranteed.

Reason: Substituting Eq. (12) into Eq. (4a) and differentiating Eq. (5b), it yields

$$\dot{S} = GCAx + GC\Delta Ax + k\operatorname{sgn}(S) + \beta\exp(-\beta t)Gy(0) \tag{15}$$

It is easy to see, if

$$k \geq \max [(\|GCA\|_2 + \|GC\|_2\|\Delta A\|_2) \|x(t)\|_2 + \beta\|Gy(0)\|_2] \quad (16)$$

then the existence condition Eq. (11b) can hold. However, $x(t)$ in Eq. (16) is not available, hence Eq. (16) should be modified as follows.

Because at the initial instant $S[y(0)] = 0$, i.e., the initial output is on the sliding hyperplane. From the concept of the equivalent motion, while in the sliding mode, we have the reduced-order system as follows with the aid of Eqs. (8a) and (8b)

$$\begin{aligned} \dot{x}_1 &= \hat{A}_1 x_1 + \phi \\ S(y) &= 0 \end{aligned} \quad (17)$$

with the initial state $x(0)$. Solving Eq. (17), we get

$$x_1(t) = \exp[\hat{A}_1(t)]\, x_1(0) + \int_0^t \exp[\hat{A}_1(t-\tau)]\phi(\tau)\, d\tau \quad (18)$$

For a stable matrix $\hat{A}_1$ and finite vector ϕ, it is clear that both $\|\exp[\hat{A}_1(t)]\|_2$ and $\|\int_0^t \exp[\hat{A}_1(t-\tau)]\phi(\tau)\,d\tau\|_2$ are limited to some bounded values. The upper bound of the state vector $x_1(t)$ is

$$\|x_1(t)\|_2 \leq \|\exp[\hat{A}_1(t)]\|_2 \|x(0)\|_2 + \left\| \int_0^t \exp[\hat{A}_1(t-\tau)]\phi(\tau)\, d\tau \right\|_2 \quad (19)$$

Further, by Eqs. (7c) and (8c), the upper bound of the state vector $x(t)$ is

$$\|x(t)\|_2 = \left\| [x_1^T(t), x_2^T(t)]^T \right\|_2 \leq \|H\|_2 \|x(t)\|_2 + \left\| (GC_2)^{-1} Gy(0) \right\|_2 \quad (20)$$

Substituting Eq. (19) into Eq. (20) and according to Eq. (16), Eqs. (13) and (14) are obtained.

Remark 2: It is noted that because $y(0)$ lies on the sliding hyperplane $S = 0$ initially, the control u in Eq. (12) with the gain k in Eq. (13) is chosen to force $y(t)$, for $t = 0 + \Delta t$, $0 < \Delta t \ll 1$, remaining on $S = 0$; moreover, each term of right-hand side of Eq. (13) is taken to be norm value and the maximum value of it is considered. Therefore, we can conclude that Eq. (13) indeed gives a strong enough gain k such that $y(t)$, $t \geq 0$, will be kept lying on the sliding hyperplane all the way. Consequently, the control selection is indeed workable.

Remark 3: Because the control u in Eq. (12) may give rise to chattering due to the term of signum vector $\mathrm{sgn}(S)$, directly applying such a control signal to the plant may be impractical. To obtain a continuous approximation control signal, each element of $\mathrm{sgn}(S)$ can be replaced by a smoothing continuous function as[17]

$$M_i(S) = \frac{s_i}{\|S\| + \delta_i}, \qquad i = 1, \ldots, m \quad (21)$$

where each $\delta_i > 0$ is a small positive constant. Because our control design only considers the existence condition of the sliding mode, the output trajectory should be confined to the neighborhood of the sliding hyperplane, so δ_i cannot be too large, $i = 1, 2, \ldots, m$. However, how to choose a set of suitable δ_i is still an open problem.

IV. Illustrative Example

It is difficult to find a practical example with multi-input system. Therefore, for convenience, we adopt the system of aircraft in Ref. 12 to be illustrated.

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\delta}_e \end{bmatrix} = \begin{bmatrix} -0.277 & 1 & -0.0002 \\ -17.1 & -0.178 & -12.2 \\ 0 & 0 & -6.67 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \delta_e \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6.67 \end{bmatrix} u \quad (22a)$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \delta_e \end{bmatrix} \quad (22b)$$

where α is the attack angle, q is the pitch rate, δ_e is the elevator angle, u is the command to the elevator, and y is the measurement

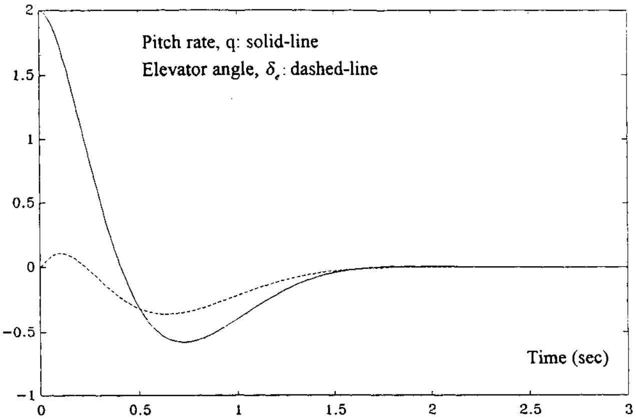

Fig. 1 Output trajectory q and δ_e.

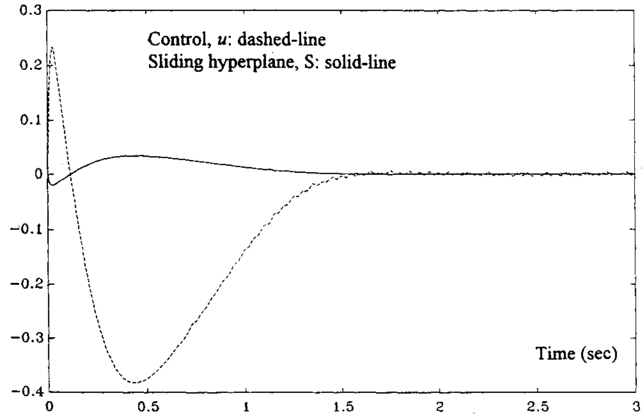

Fig. 2 Control u and the sliding hyperplane $S(y)$.

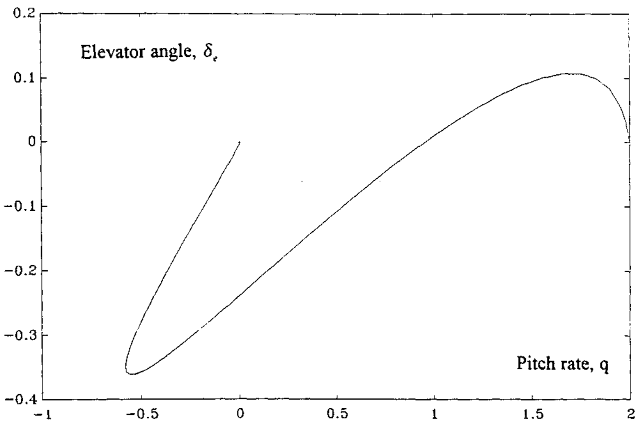

Fig. 3 Phase plane of output q and δ_e.

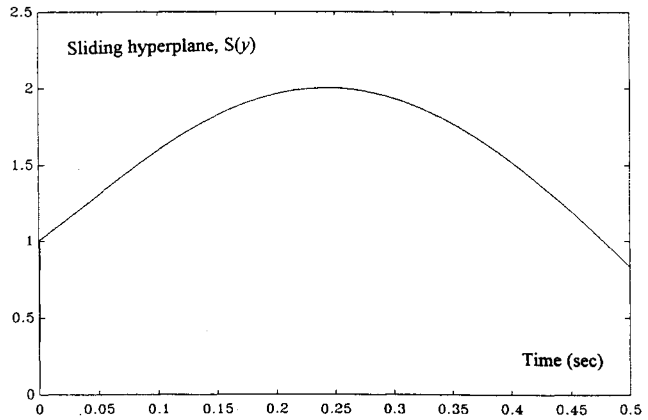

Fig. 4 The S(y) of the "Example" in Ref. 12 with $x(0) = [1\ 0\ 1]^T$.

vector. It is obvious that system Eqs. (22a) and (22b) is controllable. Suppose the uncertainty is illustrated as

$$D = 0.1\,[\sin(t) + \cos(2t)\quad \sin(3t)\cos(t)\ -\ 1 + \sin(2t)\cos(3t)] \quad (23)$$

From Eq. (23), $\|D\|_2 \le 0.2235$, and $\|\Delta A\|_2 \le 1.4910$. Hence we need not transform the system, since the matrix B is of the form in Eq. (3). Define $x_1 = [\alpha\ \ q]^T$ and $x_2 = \delta_e$, then

$$A_{11} = \begin{bmatrix} -0.277 & 1 \\ -17.1 & -0.178 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -0.0002 \\ -12.2 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$A_{21} = [-6.67], \quad B_2 = [6.67], \quad C_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $K = [k_1, k_2]$. According to lemma 1, it must satisfy

$$\text{rank}\begin{bmatrix} -1 & 0 \\ k_1 & k_2 - 1 \end{bmatrix} \le 1 \quad (24)$$

i.e., $k_2 = 1$. Hence only k_1 can affect the stability of the reduced-order system in the sliding mode. Suppose we choose $K = [-0.4635\ \ 1]$ to place the eigenvalues at $\lambda_{1,2} = -3.06 \pm 3.06j$, $j = \sqrt{-1}$. Because $K = (GC_2)^{-1}G$, it follows that $G = [-0.4635\ \ 1]$. Let $\beta = 3$ and $y(0) = [2\ \ 0]^T$, then from Eq. (6), the sliding hyperplane is

$$S(y) = -0.4635y_1 + y_2 - 2\exp(-3t) = 0 \quad (25)$$

Suppose we merely know the bound of the initial state $\|x(0)\|_2 = 2$. From Eq. (20), the upper bound $\|x(t)\|_2 \cong 5.3160$, and from Eq. (13), $k \ge 49.6299$. If we let $k = 50$, from Eq. (12), we have

$$u(y) = -7.5\,\text{sgn}(S) \quad (26)$$

Owing to remark 3, the sgn(S) in Eq. (26) can be replaced by Eq. (21) with a small δ, e.g., $\delta = 0.5$. The simulation results are shown in Figs. 1, 2, and 3, respectively. It is clear that a good approximation causes the perfect sliding motion and the chattering phenomena is eliminated.

Here, we have to mention that in the example of Ref. 12 the authors give $N = 0$ to yield

$$C^T G^T G C A = \begin{bmatrix} 0 & 0 & 0 \\ -3.6736 & -0.0382 & 0.4706 \\ 7.9259 & 0.0825 & -1.0153 \end{bmatrix}$$

which is not negative semidefinite. Therefore, the $S(y)$ trajectory with initial state $x(0) = [1\ \ 0\ \ 1]^T$ does not decrease, i.e., the reaching condition Eq. (11a) fails during a certain period of time (see Fig. 4).

V. Conclusion

If some state is unavailable and no estimated state is employed, the output feedback design in variable structure systems is difficult work. The global reaching condition is especially hard to meet. This paper has introduced a new design on the sliding hyperplane, such that the system is on the hyperplane initially. Therefore, only the existence condition of the hyperplane must be considered. Based on the concept of the equivalent motion, the constant control gain is obtained by an off-line computation with the known bound of the initial state.

Acknowledgment

Sponsorship of this research by the National Science Council of Taiwan, Republic of China, under Contract NSC 81-0404-E008-02 is deeply appreciated.

References

[1]Utkin, V. I., "Variable Structure Systems with Sliding Modes," *IEEE Transactions on Automatic Control,* Vol. AC-22, No. 2, 1977, pp. 212–222.

[2]Drazenovic, B., "The Invariance Condition in the Variable Structure Systems," *Automatica,* Vol. 5, No. 3, 1969, pp. 287–295.

[3]Young, K. D., and Kwatny, H. G., "Variable Structure Servomechanism Design and Applications to Overspeed Protection Control," *Automatica,* Vol. 18, No. 4, 1982, pp. 384–400.

[4]Lin, S. C., and Tsai, S. J., "A Microprocessor-based Incremental Servo Systems with Variable Structure," *IEEE Transactions on Industrial Electronics,* Vol. IE-31, No. 4, 1984, pp. 313–316.

[5]Harashima, F., and Harshimoto, H., "MOSFET Converter-fed Position Servo System with Sliding Mode Control," *IEEE Transactions on Industrial Electronics,* Vol. IE-32, No. 3, 1985, pp. 238–244.

[6]Bengiamin, N. N., and Chan, W. C., "Variable Structure Control of Electric Power Generation," *IEEE Transactions on Power Apparatus and System,* Vol. PAS-101, No. 2, 1982, pp. 376–380.

[7]Chan, W. C., and Hsu, Y. Y., "An Optimal Variable Structure Stabilizer for Power System Stabilization," *IEEE Transactions on Power Apparatus and System,* Vol. PAS-102, No. 6, 1983, pp. 1738–1746.

[8]Calise, A. J., and Kramer, F. S., "A Variable Structure Approach to Robust Control of VTOL Aircraft," *Journal of Guidance, Control, and Dynamics,* Vol. 7, No. 5, 1984, pp. 620–626.

[9]Bondaref, A. G., Bodaref, S. A., Kostyleva, N. E., and Utkin, V. I., "Sliding Modes in Systems with Asymptotic State Observers," *Automation and Remote Control,* Vol. 46, No. 5, 1985, pp. 679-684.

[10]Emelyanov, S. V., Korovin, S. K., Nersisian, A. L., and Nisenzon, Yu. E., "Output Feedback Stabilization of Uncertain Plants: a Variable Structure Systems Approach," *International Journal of Control,* Vol. 55, No. 1, 1992, pp. 61–81.

[11]Diong, B. M., and Medanic, J. V., "Dynamic Output Feedback Variable Structure Control for System Stabilization," *International Journal of Control,* Vol. 56, No. 3, 1992, pp. 607–630.

[12]Heck, B. S., and Ferri, A. A., "Application of Output Feedback to Variable Structure Systems," *Journal of Guidance, Control, and Dynamics,* Vol. 12, No. 6, 1989, pp. 932–935.

[13]Lukyanov, A. G. and Utkin, V. I., "Methods of Reducing Equations of Dynamic Systems to Regular Form," *Automation and Remote Control,* Vol. 42, No. 4, 1981, pp. 413–420.

[14]El-Ghezawi, O. M. E., Zinober, A. S. I., and Billings, S. A., "Analysis and Design of Variable Structure Systems Using a Geometric Approach," *International Journal of Control,* Vol. 38, No. 3, 1983, pp. 657–671.

[15]Dorling, C. M., and Zinober, A. S. I., "Two Approaches to Hyperplane Design in Multi-variable Variable Structure Control Systems," *International Journal of Control,* Vol. 44, No. 1, 1986, pp. 65–82.

[16]Davison, E. J., "On Pole Assignment in Linear Systems with Incomplete State Feedback," *IEEE Transactions on Automatic Control,* Vol. AC-15, June 1970, pp. 348–351.

[17]Burton, J. A., and Zinober, A. S. I., "Continuous Approximation of Variable Structure Control," *International Journal of Systems Science,* Vol. 17, No. 6, 1986, pp. 875–885.